# Weighted Max-Min Resource Allocation for Frequency Selective Channels

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**Abstract**

In this paper, we discuss the computation of weighted max-min rate allocation using joint TDM/FDM strategies under a PSD mask constraint. We show that the weighted max-min solution allocates the rates according to a predetermined rate ratio defined by the weights, a fact that is very valuable for telecommunication service providers. Furthermore, we show that the problem can be efficiently solved using linear programming. We also discuss the resource allocation problem in the mixed services scenario where certain users have a required rate, while the others have flexible rate requirements. The solution is relevant to many communication systems that are limited by a power spectral density mask constraint such as WiMax, Wi-Fi and UWB.

**Index Terms**

Power allocation, multi-carrier systems, rate control.

## I. Introduction

Orthogonal Frequency Division Multiple Access (OFDMA) is becoming a ubiquitous technique for wireless multiple access schemes in communication systems such as UWB, WLAN, WiMAX and LTE, due to its high spectral efficiency. OFDMA waveforms provide the flexibility of allocating subcarriers to combat frequency selective fading. These standards operate under two types of power constraints: Total power and power mask; i.e. the Power Spectral Density (PSD) of the transmitter is limited by the regulator. The total capacity of OFDMA can be optimized by dynamically allocating subcarriers among users according to channel conditions. However, the operator must satisfy the subscribers' demands to provide a reasonable level of Quality of Service (QOS). The standards define several different services that allow QOS differentiation. The major challenges facing QOS in wireless networks are the dynamic of the channels, bandwidth allocation, and handoff support. It is important to guarantee QOS at each layer so that the network stays flexible. Bandwidth and bit rates play a major role. They should be allocated in an efficient manner. In some systems data services and voice services have to be supported simultaneously. These services can conflict because voice services are very delay sensitive and require real-time service. Whereas, data services are less delay sensitive but are very sensitive to loss of data and require almost-error-free transmission. Thus both factors must be taken into account when providing QOS for voice and data services. In this paper, we address the allocation of subcarriers using a the weighted max-min approach that sets user priority according to a preset weight. This approach is then extended to guarantee a minimum data rate for voice services and allocate the residue capacity to data services.

In [1]a power adaptation method was suggested to maximize users' total data rate in downlinks of an OFDM system. The transmitted power adaptation scheme was derived by solving the maximization problem in two steps involving subcarrier assignment of users and power allocation of subcarriers. The outcome is that the data rate of a multiuser OFDM system is maximized when each subcarrier is assigned to only one user with the best channel gain for that subcarrier, and the transmit power is distributed over the subcarriers by a water-filling policy. However, fairness does not enter into this approach. In the extreme case most of the spectrum will be allocated to a small group of subscribers with high average channel gains. In [2] the problem of resource allocation of the OFDMA system was addressed. A heuristic

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scheduling algorithm was proposed under the constraint that each subscriber must obtain a preset data rate.

Rhee and Cioffi [3] derived a multiuser convex optimization problem under the total power constraint to find max-min suboptimal subcarrier allocation, where equal power is allocated to the subcarriers. A max-min rate allocation algorithm maximizes the data rate of the worst user, such that all users operate at a similar data rate. However, this solution is not suitable when the operator has to provide different level of services. Shen et al. [4] proposed a suboptimal proportional fairness resource sharing mechanism which provides multiple service levels under total power constraint while maximizing the total data rate. The algorithm involves two steps. First, the subcarriers are allocated under the assumption that the power is equal on each subcarrier. In the second step, the power is distributed among the allocated subcarriers to maximize the total rate while maintaining proportional fairness constraints. An alternative approach to the resource allocation problem is using game theoretic solutions such as the Nash bargaining solution under total power constraint (see e.g., [5], [6]), [7] or under PSD mask constraint [8] as well as the Kalai-Smorodinski solution [9], [10],[11].

Here, we focus on power spectral density maks constraint and introduce the mechanisms to enable explicit subcarrier allocation for multiple users in wireless systems when the following conditions must be fulfilled:

1) Differentiated service levels must be supported. A wireless operator should have the flexibility to specify differentiated service levels (or weights). The available radio resource has to be partitioned proportionally to the weights.
2) Voice service is supported using a fixed data rate.
3) Computational and signaling overhead must be minimal. A primary design goal of an efficient resource allocation algorithm is to minimize the communication and the computational load of feedback iterations Algorithms have to be designed to calculate the allocation that puts a minimal load on the system. Specifically, the time it takes to calculate the fair rate must be minimal.

In this paper, we show how the weighted max-min fairness design criterion can assist operators in network optimization, at multiple target rates. Here, we use a model similar to [4] but employ a power mask rather than an average power constraint. It is well known that the total data throughput of a zero-margin system is close to capacity even with a flat transmit (PSD) as long as the energy is poured only into subcarriers with high SNR gains. A good algorithm will not assign power to bad subcarriers. Furthermore, a flat PSD might be necessary if the PSD mask constraint is tighter than the total power constraint.

The remainder of the paper is organized as follows. In section II we describe the general model of the wireless system and derive a solution for the weighted max-min resource allocation problem. Section III is focused on the special solution for the case of two subscribers and outlines a simple algorithm for computing the weighted max-min solution. Simulation results are presented and discussed in Section IV. Section V concludes this paper.

## II. Resource allocation using the weighted max-min solution

In this section, we show that under a PSD mask constraint the max-min fair solution can be computed using linear programming. This is simpler than the total power constraint where general convex programming is necessary. Assume that we have $N$ users, sharing a frequency selective channel. Let the $K$ channel matrices[1] at frequencies $k = 1, ..., K$ be given by $\langle \mathbf{H}_k : k = 1, ..., K \rangle$. Each user is allowed to transmit using a maximal power $p(k)$ in the $k$'th subcarrier. In this paper, we limit ourselves to a joint FDM and TDM scheme where an assignment of disjoint portions of the frequency band to the various transmitters can be different at each time instance as is done in Wimax. In the FDM/TDM case we have the following:

1. User $n$ transmits using a PSD limited by $\langle p_n(k) : \; k = 1, ..., K \rangle$.

[1]These can be the uplink, downlink or multiple source-destination pairs within the network.

2. Each user $n$ is allocated a relative time vector $\boldsymbol{\alpha} = [\alpha_{n1}, ..., \alpha_{nK}]^T$ where $\alpha_k$ is the proportion of time allocated to user $n$ at the $k$'th frequency channel. This is the TDM/FDM part of the scheme.
3. For each $k$, $\sum_{n=1}^{N} \alpha_{nk} = 1$. This is a Pareto-optimality requirement.
4. The rate obtained by user $n$ is given by

$$R_n(\boldsymbol{\alpha}_n) = \sum_{k=1}^{K} \alpha_{nk} R_{nk}, \ . \tag{1}$$

where,

$$R_{nk} = \log_2 \left( 1 + \frac{|h_{nn}(k)|^2 p_n(k)}{\sigma_n^2(k)} \right)$$

and the subcarrier bandwidth is normalized to 1. Interference is avoided by time sharing at each frequency band; i.e, only a single user transmits at a given frequency bin at any time. Furthermore, since at each time instance each frequency is used by a single user, each user will transmit using the maximal power. *Note that we can replace the instantaneous rates by the long term averages using well known coding theorems for fading channels [12]. This allows much slower information exchange and makes the proposed approach practical in real wireless systems*.

The weighted max-min fair solution with weights $\gamma_1, ..., \gamma_N$ is given by solving the following equation:

$$R_{\max\min} = \max_{\boldsymbol{\alpha}_1,...,\boldsymbol{\alpha}_N} \min_{1 \le n \le N} \gamma_n R_n(\boldsymbol{\alpha}_n). \tag{2}$$

To solve this equation we rephrase it as a linear programming problem: Let $c$ be the value of the weighted max-min rate. We would like to maximize $c$ under the constraints $R_n \ge c$, for all $1 \le n \le N$. Since each $R_n$ depends linearly on $\boldsymbol{\alpha}_n$ we require

$$\max_{\boldsymbol{\alpha}_1,...,\boldsymbol{\alpha}_N, c} c, \tag{3}$$

under the constraints

$$\begin{array}{l} 0 \le c, \\ \frac{c}{\gamma_n} \le \sum_{k=1}^{K} \alpha_{nk} R_{nk}, \qquad n = 1, ..., N \ , \\ \sum_{n=1}^{N} \alpha_{nk} = 1, \qquad k = 1, ..., K. \end{array} \tag{4}$$

The Lagrangian is given by:

$$\begin{array}{rl} f\left(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, c\right) = & -c - \sum_{n=1}^{N} \ \delta_n \left( \sum_{k=1}^{K} \alpha_{nk} R_{nk} - c/\gamma_n \right) \\ & - \sum_{n=1}^{N} \sum_{k=1}^{K} \mu_{nk} \alpha_{nk} \\ & + \sum_{k=1}^{K} \lambda_k \left( \sum_{n=1}^{N} \alpha_{nk} - 1 \right) - \beta c. \end{array} \tag{5}$$

To better understand the problem, we first derive the KKT conditions. Taking the derivative with respect to the variables $\alpha_n(k)$ and $c$ we obtain

$$\left\{ \begin{array}{l} -\mu_{nk} + \lambda_k - \delta_n R_{nk} = 0 \\ -1 + \sum_{n=1}^{N} \frac{\delta_n}{\gamma_n} - \beta = 0, \end{array} \right. \tag{6}$$

with the complementarity conditions:

$$\left\{ \begin{array}{l} \lambda_n \left( \sum_{n=1}^{N} \alpha_{nk} - 1 \right) = 0, \\ \delta_n \left( \sum_{k=1}^{K} \alpha_{nk} R_{nk} - c/\gamma_n \right) = 0, \\ \mu_{nk} \alpha_{nk} = 0, \\ \beta c = 0, \mu_{nk} \ge 0, \beta \ge 0, \delta_n \ge 0. \end{array} \right. \tag{7}$$

Note that this problem is always feasible by choosing $c = 0$. Based on (6)-(7) we can easily see that the following proposition holds:

*Proposition 2.1:* The Lagrange multipliers in equation (7) satisfy the following claims:

1. If there is a non zero feasible solution then $\beta = 0$.
2. For each user with total rate equal to $c > 0$, $\delta_n > 0$, and $\beta = 0$. Therefore, $\sum_{n=1}^{N} \delta_n/\gamma_n = 1$. Otherwise, $\delta_n = 0$.
3. If $\alpha_{nk} > 0$, then $\mu_{nk} = 0$ and $\lambda_k = \delta_n R_{nk}$.
4. If $\alpha_{nk} = 0$, then $\mu_{nk} \geq 0$ and $\lambda_k \geq \delta_n R_{nk}$.

From these we obtain the following proposition:

*Proposition 2.2:* The weighted max-min fair solution is achieved if all users have equal weighted rates; i.e., the optimal $c$ satisfies for all $n$ $c = \gamma_n R_n$.

**Proof:** Let $c$ be the optimal value. Assume that there is a user $n$ with a rate higher than $c$ and let $k$ be a frequency such that $\alpha_n(k) > 0$. Define $\alpha'_n(k) = \alpha_n(k) - \varepsilon$, and for $m \neq n$: $\alpha'_m(k) = \alpha_m(k) + \varepsilon/(N-1)$. Obviously the weighted rate for all other users is increased. Choosing $\varepsilon \leq \gamma_n \sum_{k=1}^{K} \alpha_n(k) R_{nk} - c$, ensures that $R_n > c$. Since by construction all users $m \neq n$ achieve a rate higher than $c$ we obtain a contradiction to the optimality of $c$. This claim is important result from a network planning perspective. The achieved rates are proportional to $1/\gamma_n$; in other words, users with rates $\gamma_m, \gamma_n$ will receive rates satisfying $R_m/R_n = \gamma_m/\gamma_n$. This is desirable since utility typically scales with $\log R$, so that doubling the rate results in a fixed increase in the total utility.

## A. Voice and data rate allocation

In networks carrying mixed services, it is important to be able to allocate a fixed bandwidth, to constant-bit-rate and latency-sensitive services such as voice services. The weighted max-min formulation can be easily generalized to this case. Voice users (fixed rate) will get at least $R_{min}$, while, other variable-bit-rate users will get the weighted max-min rate according to their respective service levels. We have two groups of users: $V, D$ and the optimization becomes:

$$\max_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_N, c} c \begin{cases} 0 \leq c \\ c \leq \sum_{k=1}^{K} \alpha_{ik} R_{nk}, & i \in D \\ R_{\min} \leq \sum_{k=1}^{K} \alpha_{ik} R_{nk}, & i \in V \\ \sum_{i=1}^{N} \alpha_i(k) = 1, & k = 1, ..., K. \end{cases} \tag{8}$$

Here, one should solve the optimization problem first assuming that the set $D$ is empty. This will confirm that there is a feasible solution for the voice users. If there is a feasible solution for the set $V$ then we know that there is a feasible solution to the general problem. A simple version of this scenario is analyzed in Example II in section IV.

We now show that the feasibility of a given rate allocation can be tested by solving a simple weighted max-min problem, where the weights are given by the inverse of the desired rates. By proposition 2.2 the solution to the weighted max-min problem with weights given by $\gamma_n = 1/R_n^d$ where $R_n^d$ is the desired rate for user $n$, provides the largest $c$ such that for each user $cR_n^d = R_n$. Hence the rate vector $\left(R_1^d, ..., R_N^d\right)$ is feasible if and only if the solution satisfies $1 \leq c$. Otherwise the rate vector is infeasible. This completes the solution of the feasibility problem. Note that the solution holds even when each constant bit-rate user has a different rate requirement.

# III. THE TWO USER CASE

In this section, we address the special cases of two users. In this case the optimization problem can be dramatically simplified. Using $1-4$ in proposition 2.1 above we can easily conclude that the partition rules are as follows:

1) $\frac{\delta_1}{\gamma_1} + \frac{\delta_2}{\gamma_2} = 1$. Special case of item 2 in proposition 2.1.
2) If $\delta_1 R_{1k} > \delta_2 R_{2k}$ the frequency bin $k$ is allocated to user 1.

3) If $\delta_1 R_{1k} < \delta_2 R_{2k}$ the frequency bin $k$ is allocated to user 2.
4) If $\delta_1 R_{1k} = \delta_2 R_{2k}$ the frequency bin $k$ is shared between the users such that they both get the same total rate. based on item 3 in proposition 2.1.

An interesting consequence of our analysis is that in the two user case at most a single subcarrier should be shared between the users. This conclusion can be extended to the $N$ user case, where at most $\binom{N}{2}$ frequencies are shared in time. The proof is given in appendix I.

Based on the above properties we suggest an $O(K \log_2 K)$ complexity algorithm motivated by our analysis of the Nash Bargaining Solution (NBS) for the frequency selective interference channel [13], [8]. Extensions to the total power constraint are possible, similarly to the solution of the NBS [7]. We also show that at most a single frequency may be shared between the two users. To that end, let $\alpha_{1k} = \alpha_k$, and $\alpha_{2k} = 1 - \alpha_k$, and without loss of generality, we set $\gamma_1 = 1$, and $\gamma_2 = \gamma$. The ratio $\Gamma = \frac{\delta_2}{\delta_1} = \frac{1-\delta_1}{\delta_1 \gamma}$ is a threshold which is independent of the frequency and is set by the optimal assignment. Although $\Gamma$ is a-priori unknown, it exists. We also assume that the rate ratios $L(k) = R_{1k}/R_{2k}, 1 \le k \le K$ are sorted in decreasing order; i.e. $L(k) \ge L(k'), \forall k \le k'$. [2] Using proposition 2.2 we obtain

$$\sum_{k=1}^{K} \alpha_k R_{1k} = \gamma \sum_{k=1}^{K} \left(1 - \alpha_k\right) R_{2k}. \tag{9}$$

We are now ready to define the optimal assignment of the $\alpha_k$'s.

Let $\Gamma_k$ be a moving threshold defined by

$$\Gamma_k = \frac{A_k}{B_k \gamma} \tag{10}$$

where

$$A_k = \sum_{m=1}^{k} R_{1m}, \;\; B_k = \sum_{m=k+1}^{K} R_{2m}. \tag{11}$$

$A_k$ is a monotonically increasing sequence, while $B_k$ is monotonically decreasing. Hence, $\Gamma_k$ is also monotonically increasing. $A_k$ is the rate of user 1 respectively when frequencies $1, ..., k$ are allocated to him. Similarly $B_k$ is the rate of user 2 when frequencies $k+1, ..., K$ are allocated to him. Let

$$k_{\min} = \min_k \left\{ k : A_k \ge B_k \gamma \right\}. \tag{12}$$

We are interested in a feasible solution such that the ratio of the accumulated rate of the users will be equal to $\gamma$. Thus, frequency bin $k_{min}$ has to be split between the users, and $\alpha_{k_{min}}$ is given by

$$A_{k_{min}-1} + \alpha_{k_{min}} R_{1k_{min}} = \gamma \left( B_{k_{min}-1} - \alpha_{k_{min}} R_{2k_{min}} \right), \tag{13}$$

or

$$\alpha_{k_{min}} = \frac{\gamma B_{k_{min}-1} - A_{k_{min}-1}}{R_{1k_{min}} + \gamma R_{2k_{min}}}. \tag{14}$$

It easy to confirm that $0 \le \alpha_{k_{min}} \le 1$.

The outline of the algorithm is given in Table I.

[2]This can be achieved by sorting the frequencies according to $L(k)$.

TABLE I
ALGORITHM FOR COMPUTING THE 2X2 WEIGHTED MAX-MIN

| |
|---|
| **Initialization:** Sort the ratios $L(k)$ in decreasing order.<br>Calculate the values of $A_k, B_k$ and $\Gamma_k$. |
| Calculate $k_{min}$ using (12).<br>Calculate $\alpha_{k_{min}}$ using (13).<br>User 1 gets the bins $1 : k_{min-1}$ and $\alpha_{k_{min}}$ of bin $k_{min}$.<br>User 2 gets the bins $k_{min+1} : K$ and $1 - \alpha_{k_{min}}$ of bin $k_{min}$. |

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $R_1$ | 14 | 18 | 5 | 10 | 9 | 3 |
| $R_2$ | 6 | 10 | 5 | 15 | 17 | 16 |
| $L(k)$ | 2.33 | 1.80 | 1.00 | 0.67 | 0.53 | 0.19 |
| $A_k$ | 14 | 32 | 37 | 47 | 56 | 59 |
| $B_k$ | 63 | 53 | 48 | 33 | 16 | 0 |
| $\Gamma_k$ | .178 | .483 | .617 | 1.14 | 2.80 | $\infty$ |

TABLE II
USER RATES IN EACH FREQUENCY BIN AFTER SORTING, AND THE VALUES OF $\Gamma_k$.

## IV. EXAMPLES AND SIMULATIONS

In this section we report simulation results on rate allocation for various values of weights.

To illustrate the algorithm we compute the weighted max-min solution for the following example: *Example I*: Consider two users communicating over a 2x2 memoryless Gaussian interference channel with 6 frequency bins. The weights of user $1$ and $2$ are $1$ and $1.25$, respectively. The interference free user rates in each frequency bin (sorted according to $L_k$) are given in Table II. We now compute the values of $A_k$ and $B_k$ for each user. Since, $\Gamma_3 > 1$ we conclude that $k_{min} = 4$ and $\alpha_{k_{min}} = 0.8$. Thus, user $1$ is using subcarriers $1, 2, 3$, and sharing subcarrier $4$ with user $2$. The total rate of players $1$ and $2$ are $45$ and $36$, respectively. We can also give a geometrical interpretation to the solution. In Figure $1$ we draw the feasible total rate that player $1$ can obtain as a function of the total rate of player $2$. The enclosed area in blue, is the achievable rates set. Since, the subcarriers are sorted according to $L_k$ the set is convex. The point $(45, 36)$ is the operating point of the weight max-min with $\gamma = 1.25$. A change in the value of $\gamma$ will move the solution on the boundaries of achievable rates set.

Next, we demonstrate simulation results of rate allocation for various values of weights in two cases. In both cases the users are communicating over a frequency selective Rayleigh fading channel with variance 1. The number of frequency bins is 64 *Case 1, simulation of two data groups:* The first case simulates two groups of users, each group is of size 8. This is a typical scenario where one group has higher priority. The weight for one data group is $\gamma$ while for the second data group is $1 - \gamma$, where $0 \le \gamma \le 1$. For each value of $\gamma$ we have performed $10000$ tests. The $SNR$ values of the two data groups are 20dB and $10$ dB respectively. Figure 2 presents the distribution of the feasible rates for various value of $\gamma$. It is clear that for a given value of $\gamma$ the feasible rate will be along a ray with an angle $\phi = \arctan \frac{\gamma}{1-\gamma}$ relative to the $x$ axis. Figure 3 presents a histogram of the ray with $\gamma = 0.1$. Figure 4, presents the average value of the feasible rate for group $1$ vs. average rate of group $2$. Figure 5 shows the outage regions for outage probability of $0.1$ and $0.05$. We can clearly see that reducing the outage has significant impact on the achievable rates.

*Case 2, simulation of a voice group and two data groups:* The second case simulates three groups of users, a voice group of size 4 and two data groups each of size 8. The $SNR$ value of the voice group is $5$dB and the $SNR$ of the two data groups is $20$dB. Figure 6 shows the outage regions for outage probability of $0.05$,$0.1$ and $0.5$.

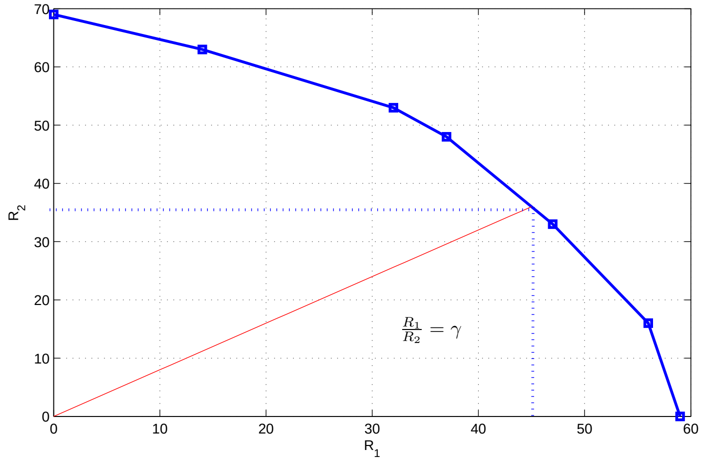


Fig. 1. The feasible total rate of player 1 vs, the feasible total rate of player 2.

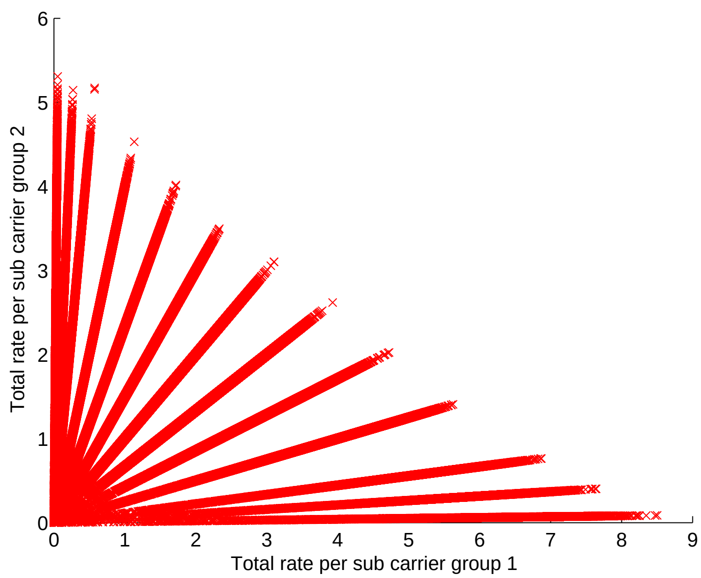


Fig. 2. The distribution of feasible rates for each value of $\gamma$. $[SNR_1, SNR_2] = [20dB, 10dB]$.

## V. Conclusion and extensions

In this paper we described a simple rate allocation technique for multiple-access OFDMA systems applying joint TDM/FDM subchannel allocation. The method is applicable whenever a central access point or base station is available. The complexity of the technique is very low. Furthermore, the allocation can be done using channel statistics instead of the actual channels. We have also demonstrated how to accommodate and test the feasibility of a set of constant rate users. Finally, we have analyzed the two user case, and provided a very low complexity weighted max-min algorithm for this case.

## VI. Appendix

**Lemma I.1 :** Assume that all the rate ratios $R_1(k)/R_2(k)$ are different from each other then at most a single frequency bin is shared between the two users.

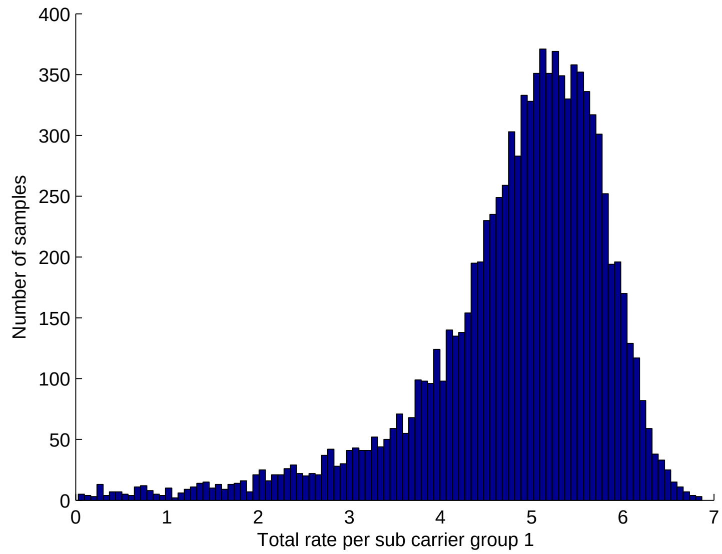


Fig. 3. A histogram of group 1 rates for $\gamma = 0.1$ and $[SNR_1, SNR_2] = [20dB, 10dB]$.

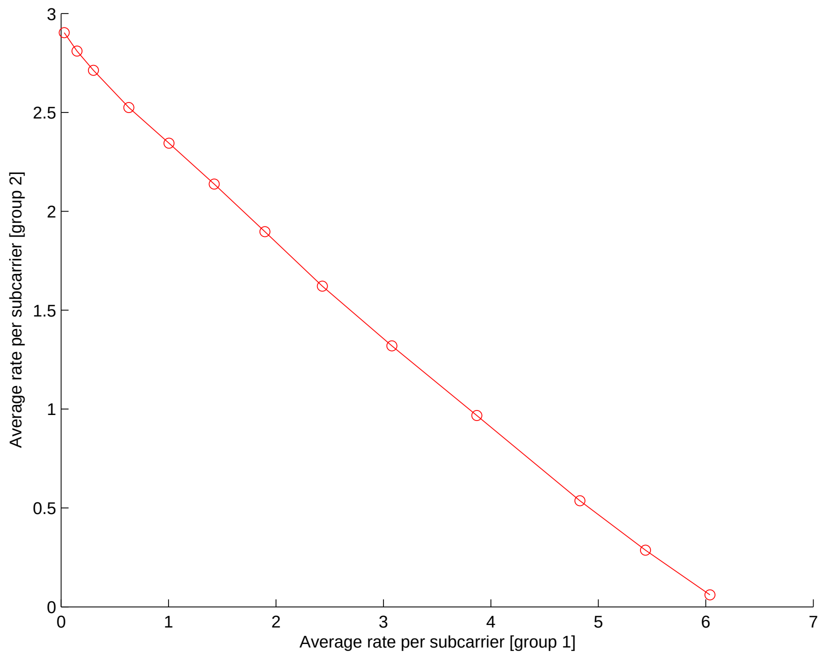


Fig. 4. The average rate of group 2 vs. the average rate of group 1 for $[SNR_1, SNR_2] = [20dB, 10dB]$.

**Proof:** Based on 3 in proposition 2.1 a subcarrier is shared between two users if $\delta_1 R_{1k} = \delta_2 R_{2k}$, or in other words $\frac{\delta_2}{\delta_1} = \frac{R_{1k}}{R_{2k}}$. Hence, if all rate ratios are different, at most a single frequency may have a rate ratio equal to $\frac{\delta_2}{\delta_1}$.

**Lemma I.2 :** Assume that there is a solution where two subcarriers are shared between the users. Then there is an alternative solution where only a single subcarrier is shared between the users.

**Proof:** Assume without loss of generality that subcarriers 1, and 2 are shared between users 1 and 2. User 1 gets fractions $\alpha_1$ and $\alpha_2$ from subcarriers 1 and 2, respectively. User 2 gets fractions $\beta_1$ and $\beta_2$ from subcarriers 1 and 2, respectively (where $\alpha_i + \beta_i = 1$). Based on proposition 2.1 the rate ratios in

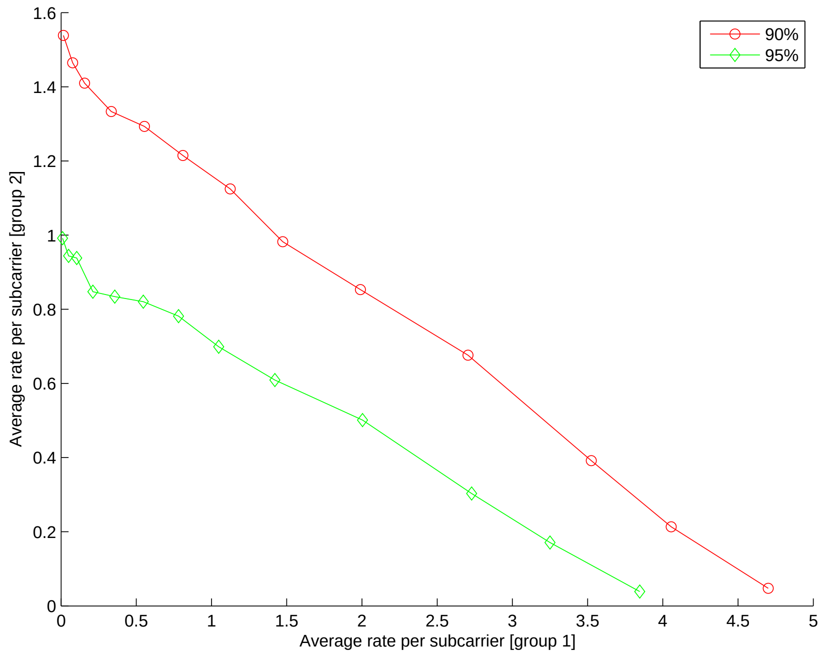


Fig. 5. The rate of group 2 vs. the rate of group 1 for outage probabilities of 10% and 5%. $[SNR_1, SNR_2] = [20dB, 10dB]$.

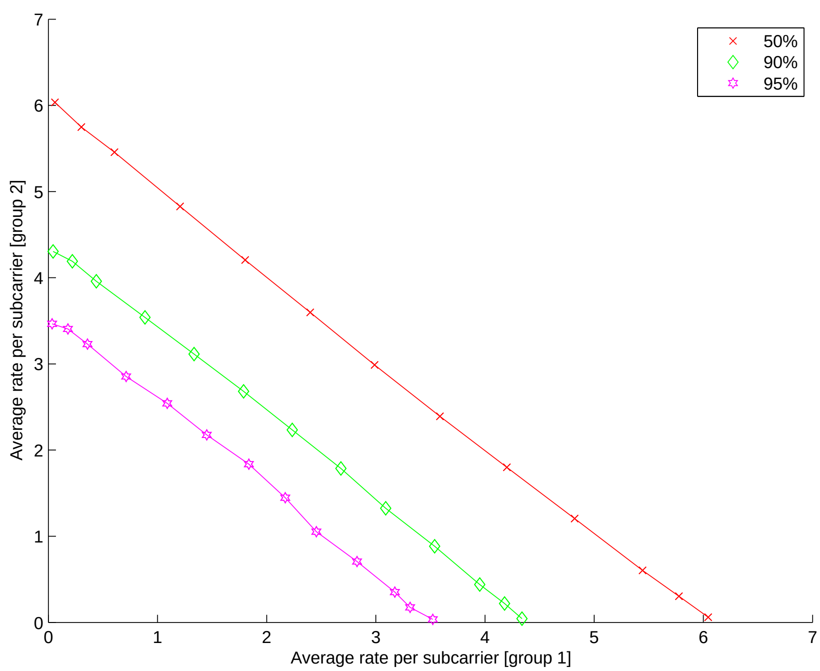


Fig. 6. The rate of data group 2 vs. the rate of data group 1 for $SNR = 20$ (voice group $SNR = 5$), and outage probabilities 0.05,0.1 and 0.5.

these frequency bins should satisfy the relation $\frac{R_{11}}{R_{21}} = \frac{R_{12}}{R_{22}}$, and the total rate of each user satisfies the conditions:

$$\begin{array}{rcl} \frac{c}{\gamma_1} & = & A + \alpha_1 R_{11} + \alpha_2 R_{12} \\ \frac{c}{\gamma_2} & = & B + \beta_1 R_{21} + \beta_2 R_{22} \end{array}, \tag{15}$$

where $A$ and $B$ are the sum of rates of users 1 and 2 on the other frequency bins. We note that in one hand, if $\alpha_1 \frac{R_{11}}{R_{12}} \leq \beta_2$, then user 1 can set $\alpha_1$ to 0 while increasing his share in subcarrier 2 by $\alpha_1 \frac{R_{11}}{R_{12}}$. On the other hand, when $\alpha_1 \frac{R_{11}}{R_{12}} > \beta_2$ we obtain $\alpha_1 > \beta_2 \frac{R_{22}}{R_{21}}$. Therefore, user 2 can set $\beta_2$ to 0 and increase his fraction in subcarrier 1 by $\beta_2 \frac{R_{22}}{R_{12}}$.

**Lemma I.3 :** In the $N$ user case at most $\binom{N}{2}$ frequencies are shared in time.

**Proof** Based on Lemma I.2. at most a single frequency bin is shared between any two users. Since the number of different pair of users is $\binom{N}{2}$, then the maximum number of frequency bins that are time shared is upper bounded by $\binom{N}{2}$.

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